Lecture notes on risk management, public policy, and the financial system

Incorporating extreme events into risk measurement

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What are stress tests?

- Stress tests analyze performance under extreme loss scenarios
- Heuristic portfolio analysis
- Steps in carrying out a stress test
  1. Determine appropriate scenarios
  2. Calculate shocks to risk factors in each scenario
  3. Value the portfolio in each scenario
- Objectives of stress testing
  - Address tail risk
  - Reduce model risk by reducing reliance on models
  - "Know the book": stress tests can reveal vulnerabilities in specfic positions or groups of positions
- Criteria for appropriate stress scenarios
  - Should be tailored to firm's specific key vulnerabilities
  - And avoid assumptions that favor the firm, e.g. competitive advantages in a crisis
  - Should be extreme but not implausible

Approaches to formulating stress scenarios

Historical scenarios based on actual past events

- Issues: time frame of individual returns, treatment of correlation
- Omitting key risk factor, such as option implied volatility for option portfolio

"What if" or **hypothetical scenarios** based on assessment of potential large market and credit events

- Must be calibrated so as to achieve appropriate severity
- May be based on models and/or macroeconomic scenario, must then be translated into asset returns

Factor-push approach in place of or (better) in addition to judgement

- Compute impact of many combinations of risk factor returns
- Stress losses define as largest portfolio losses

Strengths and weaknesses of stress tests

- Pros
  - Avoid reliance on models, model risk
  - Easy to communicate
- Cons
  - Scenarios are not directly associated with probabilities
  - Arbitrariness in scenario design
  - Difficulty including, configuring large number of risk factors

Portfolio sensitivity analysis

- Often categorized as a form of stress testing
- But focused on small changes
- Help know the book, identify concentrations, understanding drivers of P&L and risk
- Can capture nonlinearity by displaying convexities, but may miss nonlinearities that only appear if large market moves realized

Stress testing and scenario analysis

Expected shortfall

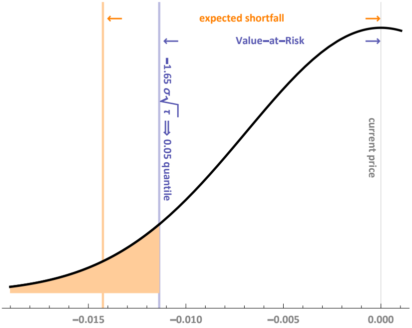
Estimating expected shortfall

Extreme value theory

Definition of expected shortfall

- **Expected shortfall** (or **conditional Value-at-Risk** or **tail Value-at-Risk** or **expected tail loss**) is defined as expected value of losses, given that VaR loss is exceeded
- Expected value of realizations of the random variable X, representing portfolio losses expressed as a positive number, in the tail of the distribution, left of the VaR scenario:

$$\mathbf{E}\left[X | X > \text{VaR}(t, \alpha, \tau)\right]$$



Expected shortfall at a 95 percent confidence level is area under density to left of VaR, divided by probability (0.05) that VaR is breached.

Estimation of expected shortfall

- VaR a *quantile* of loss distribution↔expected shortfall a *moment* of **truncated** loss distribution
- Expected shortfall for a single position can be computed using the same basic approaches used to compute VaR
  - Parametric and Monte Carlo simulation approaches can employ same specific distributional hypothesis as VaR
  - Monte Carlo and historical simulation approaches can employ same set of simulated values as VaR
- Parametric expected shortfall computed analytically
- Monte Carlo and historical simulation estimates of expected shortfall: mean of simulated losses$>$ VaR(t, α, τ)
- **Example:** 1-day expected shortfall of long position in S&P 500 index with initial value \$1 000 000 as of close on 28Aug2013

Parametric estimates of expected shortfall

- Assume returns lognormally distributed, use EWMA estimate of volatility 0.00691049 as of 28Aug2013
- Moments of truncated normal distribution can be computed analytically
- In lognormal model, ratio of expected shortfall to the VaR is

$$-\frac{\phi(z_{1-\alpha})}{(1-\alpha)z_{1-\alpha}}$$

  - $\phi(\cdot)$ represents standard normal density function, e.g. $\phi(z_{0.05}) = \phi(-1.645) = 0.103136$

conf. level	VaR	exp. shortfall	ratio
0.900	8 817.04	12 074.24	1.3694
0.950	11 302.38	14 173.64	1.2540
0.975	13 452.99	16 046.44	1.1928
0.990	15 947.66	18 270.67	1.1457

Parametric estimates of 1-day VaR and expected shortfall of $1 000 000 long position in S&P 500 on 28Aug2013.

Estimating expected shortfall via historical simulation

- Use 2 years ($T = 503$) of return observations from 28Aug2011 to 28Aug2013
- Estimated as mean of observed losses in excess of VaR scenario
- As with VaR, expected shortfall estimates may vary widely with historical look-back period

conf. level	rank	VaR	exp. shortfall	ratio
0.900	50	11 348.34	18 439.68	1.6249
0.950	25	16 147.23	23 280.36	1.4418
0.975	12	22 966.30	27 450.94	1.1953
0.990	5	26 705.46	30 872.39	1.1560

Historical simulation estimates of 1-day VaR and expected shortfall of $1 000 000 long position in S&P 500 on 28Aug2013. The rank stated in the table is that of the smallest loss included in expected shortfall and is one less than that of the VaR scenario.

Computation of expected shortfall by historical simulation

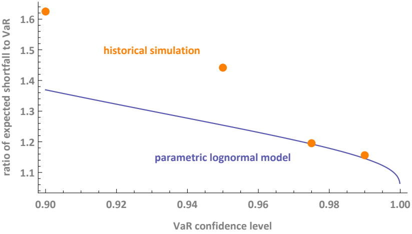
i	t	S_t	date t	S_{t-1}	$\tilde{r}^{(i)}$	$\tilde{r}^{\text{arith},(i)}$	P&L
1	52	1229.10	09Nov2011	1275.92	-0.03739	-0.03670	-36 695.09
2	18	1129.56	22Sep2011	1166.76	-0.03240	-0.03188	-31 883.16
3	17	1166.76	21Sep2011	1202.09	-0.02983	-0.02939	-29 390.48
4	25	1099.23	03Oct2011	1131.42	-0.02886	-0.02845	-28 450.97
5	46	1218.28	01Nov2011	1253.30	-0.02834	-0.02794	-27 942.23
6	9	1154.23	09Sep2011	1185.90	-0.02707	-0.02671	-26 705.46
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The entries in the penultimate column are the ordered arithmetic historical returns $\tilde{r}^{(i)}, i = 1, \ldots, T$ and $T = 503$. The P&L realizations are $x\left(\tilde{S}^{(i)} - S_t\right)$.

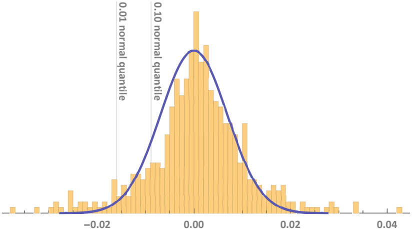
Relationship of expected shortfall to VaR

- Expected shortfall is always at least as great as the VaR: average of loss levels greater than the VaR
  - Can be much larger than VaR if return distribution is heavy-tailed and skewed
- Ratio of expected shortfall to VaR is higher at lower confidence levels and falls toward 1 for very high confidence levels
  - Many large observations beyond VaR at lower confidence level
- The normal distribution is thin-tailed$\rightarrow$
  - Parametric estimates of ratio of expected shortfall to VaR relatively low at lower confidence level
  - Highlights disadvantage of standard model: implies risk of very large losses relatively low
- Empirical distributions heavy-tailed$\rightarrow$
  - Historical simulation expected shortfall generally quite high relative to VaR
  - Disparity shrinks as confidence level rises

Estimation of expected shortfall by historical simulation



Ratio of expected shortfall to VaR as a function of confidence level.



Comparison of empirical distribution of log S&P returns 28Aug2011 to 28Aug2013 and normal model.

Advantages and disadvantages of expected shortfall

- Expected shortfall in principle oriented toward tail-risk measurement
  - More appropriate than VaR for use in setting (→)**economic capital**
  - States average extreme event if extreme event should occur
- But unlikely *per se* to provide significant improvement in tail risk measurement
  - Not really an *alternative* to VaR
  - Alternative statistic *within* VaR framework
  - If data and model not providing good tail risk estimate, expected shortfall won't help much

Backtesting expected shortfall is difficult

- Difficult to backtest, since extremes by definition more infrequent than observations in the body of the distribution
- VaR backtesting involves counting episodes in which a quantile is exceeded over some past period
- VaR ES involves counting episodes in which a conditional mean size of exceedance on each date is exceeded over some past period
    - At any useful confidence level, relatively few exceedances of VaR
    - Each one provides one observation on exceedance size
    - But you need many to have an estimate of the conditional mean
- In addition to limitations of VaR backtesting
- Revised Basel standards backtest VaR at 97.5- and 99-percent confidence levels as internal model check
    - No requirement to test ES itself

The elicitability problem

- **Elicitability** is a desirable property of a statistic of a random variable such as P&L
- Forecasts of elicitable statistics can be tracked day by day to see how close they are to their realizations (**scoring**)
- VaR is elicitable using this **scoring function:**
  - Each day, measure the absolute value of the difference between VaR and P&L
  - Weight is α if there is an excession, otherwise $1 - \alpha$
  - A low score indicates few excessions, thus validates VaR
  - But note that a zero score would indicate P&L has no variability
- ES is not elicitable, i.e. no such scoring function can be formulated
- But ES can nonetheless be evaluated through backtesting

Modeling extremes

- **Extreme value theory:** (EVT) set of models focused on extreme events rather than entire return distribution
- Losses a random variable $\tilde{x}$ with observed realizations
  - Treat losses as positive numbers for convenience
  - E.g. multiply returns by -1 for long position
- Goals of EVT
  - Extract information about extreme losses that go beyond what has been observed empirically
  - Determine probability distributions of extremes for application in risk analysis
- What is an extreme? Two standard definitions

  Block maximum: maximum value in a set of successive observations over a given time horizon: $\max(X_1, X_2, \ldots X_T)$

  Peaks over threshold: realizations exceeding a given high threshold: $\{X_t | X_t > u\}$, with u a "large number"

Extreme value distributions

- Probability distributions of block maxima converge to **generalized extreme value** (GEV) **distribution** as $T \to \infty$
  - Few assumptions regarding distribution, just i.i.d. from *some distribution*
  - Extreme value must be normalized in some way
- Analogous to **central limit theorem** (CLT):
  - Sum of independent random variates with finite mean and variance converges$\rightarrow$normal distribution
  - CLT applies to normalized random variates
- GEV distribution comes in 3 variants
  - Thin, e.g. mortality: no possibility of exceeding some finite limit
  - "Normal"
  - Fat-tailed, e.g. most asset returns: low but material probability of very large (loss) realization

Power laws and the tail index

- If a GEV-distributed random variable falls into fat-tailed category, then large losses follow a **power law**:

$$\mathbf{P}\left[X_t \geq x\right] = L(X_t)X_t^{-\iota}, \quad x, \iota > 0,$$

  - $L(X)$ is a normalizing function that varies little with t
  - For example, a constant or the logarithmic function, which rises very gradually for large values of the argument
- ι called the **tail index**.
  - Fat-tailed asset returns/losses have tail index in excess of 2

Estimating the tail index

- Simple approach: **Hill's estimator**
- To apply Hill's estimator to loss on long position in single asset
  - Multiply set of returns by -1
  - Set (somewhat arbitrarily) threshold u for extreme loss

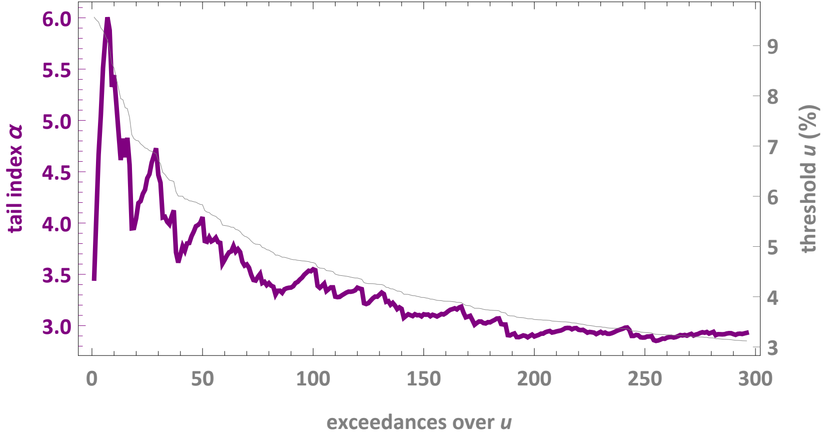
$$\{X_t\}_{t=1,\ldots,T} = \{-r_t | r_t < -u\}$$

  - Or, equivalently, include k largest losses in the data set: largest k order statistics, denoted $\{X_{(1)},\ldots,X_{(k)}\}$, with $X_{(2)} \geq \ldots \geq X_{(k)}$
- Estimator is the reciprocal of the mean log excess of losses over the threshold u or $X_{(k)}$:

$$\hat{\iota} = \left[\frac{1}{k}\sum_{j=1}^{k} \ln(X_{(j)}) - \ln(X_{(k)})\right]^{-1}$$

- **Example:** S&P 500 daily returns since 1928
  - Bad news: varies considerably with "user input," the threshold
  - Good news: converges to about 3.0

Estimating the tail index



Estimates of the tail index ι using Hill's estimator and k largest-magnitude negative returns, with $k = 5, \ldots, 300$. Purple plot: $\hat{\iota}$;gray plot: k. *Data source*: Bloomberg Financial L.P.